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Lipschitz Spaces of Holomorphic Functions over Bounded Symmetric Domains in C^N

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INTRODUCTION

T. M. Flett [7] introduced the Lipschitz spaces $HA(\alpha, p, q)$ of holomorphic function f on the unit disc such that

$$\|f\|_{\alpha,p,q} = \left[\int_0^1 (1-r)^{q-1} \left(\int_0^{2\pi} |J^{-\alpha-1} f(re^{i\theta})|^p \, d\theta\right)^{q/p} dr\right]^{1/q} < \infty$$

where $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, α is a number, and J^α is a multiplier transformation. These are spaces which generalize the classical Lipschitz spaces. The spaces $HA(\alpha, \infty, \infty)$ on the unit disc for $0 < \alpha < 1$ are the classical Lipschitz spaces of functions holomorphic on the disc and continuous on the boundary that satisfy a Lipschitz condition of order α. For any real numbers α, β the space $H \wedge (\alpha, p, q)$ is equivalent to the space $HA(\beta, p, q)$. For $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $HA(\alpha, p, q)$ is a Banach space with the norm $\| \; \|_{\alpha,p,q}$. Moreover, the dual space of $HA(\alpha, p, q)$ is equivalent to the space $HA(-\alpha, p', q')$ for $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and is equivalent to the space $HA(1/p - 1 - \alpha, \infty, \infty)$ for $0 < p < 1$, $0 < q < 1$. In particular, by the work of Duren *et al.* [4], the dual space of H^p is equivalent to the space $HA(1/p - 1, \infty, \infty)$ for $0 < p < 1$.

In this paper we extend these results to bounded symmetric domains in $\mathbb{C}^N$. Section 1 gives the necessary definition and some results in [3]. In Section 2 we define the Lipschitz spaces $HA(\alpha, p, q)$ over bounded symmetric domains in $\mathbb{C}^N$. Then we prove some properties of these spaces and an inclusion relation between H^p and $HA(\alpha, p, q)$. In Section 3 the connection between the classical Lipschitz spaces and the space $HA(\alpha, p, q)$ is discussed. We give a necessary and sufficient condition that f belongs to the classical Lipschitz class A_α in $\mathbb{C}^N$ and prove that $A_\alpha = HA(\alpha/N, \infty, \infty)$ for $0 < \alpha < 1$. In Section 4 we consider the dual space $HA^*(\alpha, p, q)$ of $HA(\alpha, p, q)$. We define continuous bilinear forms on $HA(\alpha, p, q)$ and

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$HA(-\alpha, p', q')$ and give the representation for continuous linear functionals on $HA(\alpha, p, q)$. In the last section we take the domain D to be the ball in $\mathbb{C}^N$ and prove that the dual space of $HA(\alpha, p, q)$ is equivalent to the space $HA(1/p - 1 - \alpha, \infty, \infty)$ for $0 < p \leqslant 1$, $0 < q \leqslant 1$ and that the dual space of H^p is equivalent to the space $HA(1/p - 1, \infty, \infty)$ for $0 < p < 1$.

1. Definitions and Preliminary Results

Let C^N be the vector space of N complex dimensions ($N > 1$). A domain D in C^N is homogeneous if the group G of holomorphic automorphisms of D is transitive; that is, for any pair of points $z, w \in D$ there exists a $g \in G$ such that $w = g(z)$ [9, p. 312]. A symmetric domain D is a homogeneous domain such that for any $w \in D$ there exists a $g \in G$ which satisfies the following conditions: (1) $g(w) = w$, but $g(z) \neq z$ for every $z \neq w$ of D; (2) g is an idempotent in the group G [9, p. 313]. For a bounded symmetric domain D in C^N, where we assume $0 \in D$, it is known that D is circular and starshaped with respect to the origin 0; that is, $tz \in D$ when $z \in D$ and $t \in C$ with $|t| \leqslant 1$ [13]. It has a Bergman–Šilov boundary b, which is also circular and invariant under G [13]. The isotropy group $G_0 = (g \in G : g(0) = 0)$ of G is transitive on b, and b has a unique normalized G_0-invariant measure $V^{-1}\, ds_t$, where V is the euclidean volume of b and ds_t is the euclidean volume element at $t \in b$.

Let D be a bounded symmetric domain in C^N and $0 \in D$. For any complex-valued function f holomorphic on D, and for $0 < r < 1$, we set

$$M_p(f, r) = \left(\frac{1}{V}\int_b |f(rt)|^p\, ds_t\right)^{1/p} \qquad (0 < p < \infty),$$
$$M_\infty(f, r) = \sup_{t \in b} |f(rt)|, \tag{1}$$
$$\|f\|_p = \sup_{0 \leqslant r < 1} M_p(f, r) = \lim_{r \to 1-} M_p(f, r).$$

The class of such functions f with $\|f\|_p < \infty$ is the Hardy space $H^p(D)$. For $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, $\alpha > 0$ we define

$$\begin{aligned} N_{p,q,\alpha}(f) &= \left(\int_0^1 (1 - r)^{Nq\alpha - 1} M_p^q(f, r)\, dr\right)^{1/q} \qquad (0 < q < \infty), \\ N_{p,\infty,\alpha}(f) &= \sup_{0 \leqslant r < 1} ((1 - r)^{N\alpha} M_p(f, r)). \end{aligned} \tag{2}$$

This expression $N_{p,q,\alpha}(f)$ can be regarded as a measure of the rate of growth of $M_p(f, r)$ when $M_p(f, r)$ is unbounded.

If $p, q \geqslant 1$, using Minkowski's inequality twice, we have

$$N_{p,q,\alpha}(f+g) \leqslant N_{p,q,\alpha}(f) + N_{p,q,\alpha}(g). \tag{3}$$

Similarly, if $\min(p,q) = s < 1$, using Minkowski's inequality and the inequality $(a+b)^p \leqslant a^p + b^p$ for $a, b \geqslant 0$ and $0 < p < 1$ we have

$$N^s_{p,q,\alpha}(f+g) \leqslant N^s_{p,q,\alpha}(f) + N^s_{p,q,\alpha}(g). \tag{4}$$

By [3, Lemmas 1 and 2] we have following results.

THEOREM A. *Let $0 < p < s \leqslant \infty$, $0 < q < t \leqslant \infty$, $\alpha > 0$, $\delta = 1/p - 1/s$, and f be holomorphic on the bounded symmetric domain D in C^N such that $N_{p,q,\alpha}(f) < \infty$. Then*

$$N_{p,t,\alpha}(f) \leqslant BN_{p,q,\alpha}(f),$$
$$N_{s,q,\alpha+\delta}(f) \leqslant BN_{p,q,\alpha}(f)$$

and

$$M_p(f,r) = o((1-r)^{-N\alpha}) \qquad as \quad r \to 1.$$

Let $z_{k\nu}$ denote the monomial $z_1^{\nu 1} \cdots z_N^{\nu N}$ of degree $k = \sum_{j=1}^N \nu_j$, $k = 0, 1, 2, \ldots$, $\nu = 1, 2, \ldots, m_k = \binom{N+k-1}{k}$. From the set $\{z_{k\nu}\}$ Hua constructed by group representation theory a system $\Phi_0 = \{\phi_{k\nu}\}$ of homogeneous polynomials; $\phi_{k\nu}$ is a homogeneous polynomial of degree k for all ν. It is complete and orthogonal on D, and can be made orthonormal on b [12]. Any holomorphic function on D has a Fourier series expansion [11],

$$f(z) = \sum_{k,\nu} A_{k\nu}(f)\,\phi_{k\nu}(z),$$
$$A_{k\nu}(f) = \lim_{r\to 1}(f_r, \phi_{k\nu}), \qquad (f_r, \phi_{k\nu}) = \int_b f(rt)\,\overline{\phi_{k\nu}(t)}\,ds_t, \tag{5}$$

where

$$\sum_{k,\nu} = \sum_{k=0}^{\infty} \sum_{\nu=1}^{m_k}, \qquad m_k = \binom{N+k-1}{k}.$$

Series (5) converges uniformly on compact subsets of D.

We define the multiplier transformation $J^\alpha f$ of f, where α is real number, by

$$J^\alpha f(z) = \sum_{k,\nu} (k+1)^{-\alpha} A_{k\nu}(f)\,\phi_{k\nu}(z), \qquad z \in D, \tag{6}$$

where f is given by (5). It may be regarded as a fractional integral for $\alpha > 0$, or a fractional derivative for $\alpha < 0$. Obviously,

$$J^{\alpha}(J^{\beta}f) = J^{\alpha+\beta}f,$$

for all α, β; moreover, for any positive integer m it is easy to show that

$$J^{-m}f(wt) = \left(\frac{\partial}{\partial w} w\right)^m f(wt), \qquad w \in C, \quad |w| < 1, \quad t \in b. \tag{7}$$

In particular, if $m = 1$

$$J^{-1}f(wt) = \frac{\partial}{\partial w}(wf(wt)).$$

There is also an integral formula for $J^{\alpha}f$ when $\alpha > 0$ [3], namely,

$$J^{\alpha}f(rt) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\log \frac{1}{x}\right)^{\alpha-1} f(rxt)\, dx, \qquad 0 \leqslant r < 1. \tag{8}$$

The function $J^{\alpha}f$ is clearly homomorphic on D for all real α [3].

For later reference we now list some known results. Theorem B is due to Mitchell and Hahn [14, Theorem 4] and is a generalization of a theorem of Hardy and Littlewood relating to bounded symmetric domains in C^N. Theorems C–F are due to the author [3].

THEOREM B. *Let* $0 < p < s \leqslant \infty$, $p \leqslant q < \infty$, $\delta = 1/p - 1/s$, *and* $f \in H^p(D)$, *where* D *is a bounded symmetric domain in* C^N. *Then*

$$N_{s,q,\delta}(f) \leqslant B\, \|f\|_p, \tag{9}$$

and

$$M_s(f, r) \leqslant B\, \|f\|_p (1-r)^{-N\delta}. \tag{10}$$

In (10) *large* O *can be replaced by small* o [3], *that is,*

$$M_s(f, r) = o((1-r)^{-N\delta}) \qquad \text{as} \quad r \to 1. \tag{11}$$

THEOREM C. *Suppose that* D *is a bounded symmetric domain in* C^N. *Let* p, q, s *satisfy one of the following conditions*:

(a) $0 < p < s < \infty$, $0 < q \leqslant s$, $s > 1$;

(b) $0 < p \leqslant s \leqslant \infty$, $0 < q \leqslant 1 \leqslant s$;

(c) $0 < p \leqslant s < 1$, $0 < q \leqslant s$.

Let also $\alpha > 0$, $\delta = 1/p - 1/s$, *and* f *be holomorphic on* D *such that* $N_{p,q,\alpha}(J^{N(-\alpha-\delta)}f) < \infty$. *Then* $f \in H^s(D)$ *and*

$$\|f\|_s \leqslant BN_{p,q,\alpha}(J^{N(-\alpha-\delta)}f). \tag{12}$$

Moreover, if $p = \infty$, *then* f *has a continuous extension to the Bergman–Šilov boundary* b *of* D.

THEOREM D. *Suppose that* D *is a bounded symmetric domain in* C^N. *Let* $0 < p, q \leqslant \infty$, $\beta > 0$, $\gamma > 0$, *and* f *be a holomorphic function on* D. *Then*

$$BN_{p,q,\beta}(f) \leqslant N_{p,q,\gamma}(J^{N(\beta-\gamma)}f) \leqslant BN_{p,q,\beta}(f). \tag{13}$$

THEOREM E. *Suppose that* D *is a bounded symmetric domain in* C^N. *If* $0 < p < s < \infty$, $\beta = 1/p - 1/s$, *and* $f \in H^p(D)$. *Then* $J^{N\beta}f \in H^s(D)$ *and*

$$\|J^{N\beta}f\|_s \leqslant B\,\|f\|_p. \tag{14}$$

THEOREM F. *Let* $0 < p \leqslant \infty$, *and* $\alpha > 0, f \in H^p(D)$. *Then for* $0 \leqslant r < 1$

$$M_p(J^{-N\alpha}f, r) \leqslant B(1-r)^{-N\alpha}\,\|f\|_p. \tag{15}$$

2. THE LIPSCHITZ SPACES $H\Lambda(\alpha, p, q)$ AND $H\lambda(\alpha, p, \infty)$

Motivated by the work of Flett [7], we define the Lipschitz spaces over bounded symmetric domains in C^N as follows.

Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, *and* α *be real. Then* $H\Lambda(\alpha, p, q)$ *is the complex linear space of holomorphic functions* f *on* D *such that*

$$\|f\|_{\alpha,p,q} = N_{p,q,1}(J^{N(-\alpha-1)}f) < \infty. \tag{16}$$

If $s = \min(p, q) \geqslant 1$, then the triangle inequality $\|f+g\|_{\alpha,p,q} \leqslant \|f\|_{\alpha,p,q} + \|g\|_{\alpha,p,q}$ follows for $f, g \in H\Lambda(\alpha, p, q)$ by formula (3), and $\|\ \|_{\alpha,p,q}$ is a norm on $H\Lambda(\alpha, p, q)$. If $s < 1$, then by formula (4), $\|f+g\|^s_{\alpha,p,q} \leqslant \|f\|^s_{\alpha,p,q} + \|g\|^s_{\alpha,p,q}$, and

$$d(f, g) = \|f-g\|^s_{\alpha,p,q} \tag{17}$$

is a translation invariant metric on $H\Lambda(\alpha, p, q)$. From Theorem D we see that $N_{p,q,1}(J^{N(-\alpha-1)}f) \simeq N_{p,q,-\alpha}(f)$ for $\alpha < 0$, where the symbol "$\simeq$" means norm equivalence.

For $0 < p \leqslant \infty$ *and* α *real we define the space* $H\lambda(\alpha, p, \infty)$ *to be the subspace of* $H\Lambda(\alpha, p, \infty)$ *for which*

$$M_p(J^{N(-\alpha-1)}f, r) = o((1-r)^{-N}) \qquad \text{as} \quad r \to 1^-. \tag{18}$$

In the next theorem we prove the completeness of $H\Lambda(\alpha, p, q)$ and $H\lambda(\alpha, p, \infty)$. First, we prove the following lemma.

LEMMA 1. *Let* D *be a bounded symmetric domain; suppose that* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, α *real and* $f \in H\Lambda(\alpha, p, q)$. *Then*

$$\text{(i)} \quad |J^{N(-\alpha-1)}f(z)| \leqslant B(1-r)^{N(-1-1/p)} \|f\|_{\alpha,p,q}, \tag{19}$$

for $z \in \bar{D}_r$, *where* B *is independent of* f *and* $\bar{D}_r$ *is the closure of* $\{rz : z \in D\}$.

$$\text{(ii)} \quad |J^{N(-\alpha-1)}f(z)| \leqslant C \|f\|_{\alpha,p,q}, \tag{20}$$

for $z \in K$, *a compact subset of* D, *where* C *depends on* K, p, q *and* α, *but not on* f. *Hence if* $\|f - f_n\|_{\alpha,p,q} \to 0$ *as* $n \to \infty$, *then* $J^{N(-\alpha-1)}f_n \to J^{N(-\alpha-1)}f$ *uniformly on a compact subset of* D.

Proof. By Theorem A for $p < s = \infty$, $q < t = \infty$ (for $p = q = \infty$ by definition of $N_{\infty,\infty,1}(J^{N(-\alpha-1)}f)$), we have

$$N_{\infty,\infty,1+1/p}(J^{N(-\alpha-1)}f) \leqslant BN_{p,q,1}(J^{N(-\alpha-1)}f) = B\|f\|_{\alpha,p,q},$$

and the definition of $N_{\infty,\infty,1+1/p}(J^{N(-\alpha-1)}f)$ gives

$$\sup_{t \in b} |J^{N(-\alpha-1)}f(rt)| \leqslant B(1-r)^{N(-1-1/p)} \|f\|_{\alpha,p,q},$$

and (19) follows. Now any compact subset K is contained in $\bar{D}_{r_0}$ for some r_0, $0 < r_0 < 1$. Let $C = \sup_{0<r<r_0} B(1-r)^{N(-1-1/p)}$. Then (20) follows for all $z \in K$.

THEOREM 1. *Let* D *be a bounded symmetric domain. Suppose that* $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ *and* α *is real. Then the spaces* $H\Lambda(\alpha, p, q)$ *and* $H\lambda(\alpha, p, \infty)$ *are complete and hence Banach spaces if* $p, q \geqslant 1$. *For other* p, q *these spaces are complete linear Hausdorff spaces. Furthermore,* $H\lambda(\alpha, p, q)$ *is closed in* $H\Lambda(\alpha, p, q)$.

Proof. Let $\{f_n\}$ be a Cauchy sequence in $H\Lambda(\alpha, p, q)$. Then by Lemma 1, the sequence $\{J^{N(-\alpha-1)}f_n\}$ converges uniformly on compact subsets of D to a function g, which is holomorphic on D. Let $f = J^{N(\alpha+1)}g$. Then f is holomorphic on D and $J^{N(-\alpha-1)}f=g$. For $0\leqslant r<1$, $0<p\leqslant\infty$, by uniform convergence we have

$$M_p(J^{N(-\alpha-1)}f, r)=\lim_{n\to\infty} M_p(J^{N(-\alpha-1)}f_n, r).$$

By the same methods as those in [7], for $q<\infty$, by Fatou's lemma

$$\begin{aligned}\|f\|^q_{\alpha,p,q}&=\int_0^1 \lim_{n\to\infty}(1-r)^{Nq-1}M_p^q(J^{N(-\alpha-1)}f_n, r)\,dr\\&\leqslant \varliminf_{n\to\infty}\|f_n\|^q_{\alpha,p,q}.\end{aligned}$$

Similarly the result follows if $q=\infty$. Hence $f\in H\Lambda(\alpha,p,q)$.

Further, for $m=1,2,3,\ldots$ and fixed $r<1$, we have

$$M_p(J^{N(-\alpha-1)}(f-f_m), r)=\lim_{n\to\infty} M_p(J^{N(-\alpha-1)}(f_n-f_m), r),$$

since the sequence is uniformly convergent on compact subsets of D. By the same argument as above

$$\|f-f_m\|_{\alpha,p,q}\leqslant \varliminf_{n\to\infty}\|f_n-f_m\|_{\alpha,p,q},$$

where the right side of the last inequality tends to 0 as $m\to\infty$. Hence $\{f_m\}$ converges to f in $H\Lambda(\alpha,p,q)$. This proves that $H\Lambda(\alpha,p,q)$ is complete.

Let $\{g_n\}$ be a sequence in $H\lambda(\alpha,p,\infty)$ converging to g. We show that $g\in H\lambda(\alpha,p,\infty)$. Since $H\lambda(\alpha,p,\infty)\subset H\Lambda(\alpha,p,\infty)$ and $H\Lambda(\alpha,p,\infty)$ is complete, we have $\|g_n-g\|_{\alpha,p,\infty}\to 0$ as $n\to\infty$ and $M_p(J^{N(-\alpha-1)}g_n, r)^N(1-r)^N\to 0$ as $r\to 1^-$. Also

$$\begin{aligned}&(1-r)^{Np}M_p^p(J^{N(-\alpha-1)}g, r)\\&\qquad\leqslant [(1-r)^{Np}M_p^p(J^{N(-\alpha-1)}(g_n-g), r)+(1-r)^{Np}M_p^p(J^{N(-\alpha-1)}g_n, r)],\end{aligned}$$

for n sufficiently large and fixed, and for r sufficiently close to 1. Hence

$$(1-r)^N M_p(J^{N(-\alpha-1)}g, r)\to 0 \qquad \text{as} \quad r\to 1,$$

so that $g\in H\lambda(\alpha,p,\infty)$. Thus $H\lambda(\alpha,p,\infty)$ is closed in $H\Lambda(\alpha,p,\infty)$, and $H\lambda(\alpha,p,\infty)$ is complete also.

THEOREM 2. *Let D be a bounded symmetric domain, and f be holomorphic on D, and for $0<r<1$, $f_r(z)=f(rz)$. Then*

(i) *$f_r \in HA(\alpha, p, q)$ for all $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and α real; in particular, also $f_r \in H\lambda(\alpha, p, \infty)$. If $f \in HA(\alpha, p, q)$, then $\|f_r\|_{\alpha,p,q} \leqslant B\|f\|_{\alpha,p,q}$.*

(ii) *If $q < \infty$, and $f \in HA(\alpha, p, q)$ for $0 < p \leqslant \infty$, then $f_r \to f$ in $HA(\alpha, p, q)$ as $r \to 1^-$.*

(iii) *If $q = \infty$ and $f \in HA(\alpha, p, \infty)$ for $0 < p \leqslant \infty$, then $f_r \to f$ in $HA(\alpha, p, \infty)$ if and only if $f \in H\lambda(\alpha, p, \infty)$.*

Proof. Part (i) is trivial, since for fixed $r < 1, f_r$ is holomorphic on $\bar{D}$, and $\|f_r\|_{\alpha,p,q} \leqslant \|f\|_{\alpha,p,q}$ by the increasing property of the mean M_p.

(ii) For each fixed $\rho < 1$, since $J^{N(-\alpha-1)}f$ is uniformly continuous on the compact set $\bar{D}_\rho$, then

$$\lim_{r \to 1^-} M_p(J^{N(-\alpha-1)}(f_r - f), \rho) = 0. \tag{21}$$

Also

$$M_p^s(J^{N(-\alpha-1)}(f_r - f), \rho) \leqslant 2M_p^s(J^{N(-\alpha-1)}f, \rho),$$

where $s = \min(p, 1)$. Hence

$$M_p(J^{N(-\alpha-1)}(f_r - f), \rho) \leqslant BM_p(J^{N(-\alpha-1)}f, \rho). \tag{22}$$

Since $f \in HA(\alpha, p, q)$, then by (21), (22) and Lebesgue's dominated convergence theorem, $\lim_{r \to 1^-} \|f_r - f\|^q_{\alpha,p,q} = 0$.

(iii) Now $f_r \in H\lambda(\alpha, p, \infty)$ for $0 < r < 1$, and $H\lambda(\alpha, p, \infty)$ is closed in $HA(\alpha, p, \infty)$ by Theorem 1. Hence if $f_r \to f$ in $HA(\alpha, p, \infty)$, then $f \in H\lambda(\alpha, p, \infty)$.

Conversely, if $f \in H\lambda(\alpha, p, \infty)$, multiply both side of (22) by $(1-\rho)^N$. Then for any $\varepsilon > 0$, there exists ρ_0, $0 < \rho_0 < 1$, such that

$$(1-\rho)^N M_p(J^{N(-\alpha-1)}(f_r - f), \rho) < \varepsilon$$

if $\rho_0 < \rho < 1$ and for all $r \in (0, 1)$.

For fixed ρ_0, $\bar{D}_{\rho_0}$ is a compact subset of D, and $J^{N(-\alpha-1)}f$ is uniformly continuous on $\bar{D}_{\rho_0}$. Then by (21) we can choose r_0 sufficiently close to 1 such that

$$(1-\rho)^N M_p(J^{N(-\alpha-1)}(f_r - f), \rho) < \varepsilon$$

if $0 \leqslant \rho \leqslant \rho_0$, $r_0 < r < 1$. Hence $\|f_r - f\|_{\alpha,p,\infty} < \varepsilon$ if $r > r_0$. Thus

$$\lim_{r \to 1^-} \|f_r - f\|_{\alpha,p,\infty} = 0.$$

From Theorem D with $\beta = 1$ and f replaced by $J^{N(-\alpha-1)}f$, we have the following result.

THEOREM 3. *Let D be a bounded symmetric domain; let f be holomorphic on D; and let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$ and α be real. Then $f \in H\Lambda(\alpha, p, q)$ if and only if there exists $\gamma > 0$ such that*

$$N_{p,q,\gamma}(J^{N(-\alpha-\gamma)}f) < \infty.$$

The multiplier transformation J^{α} is one to one on $H(D)$ of the space of all holomorphic functions on D, and the inverse of J^{α} is $J^{-\alpha}$. Since $J^{N(\beta-\alpha)}$ is one to one from $H\Lambda(\alpha, p, q)$ onto $H\Lambda(\beta, p, q)$ with inverse $J^{N(\alpha-\beta)}$, and $\|J^{N(\beta-\alpha)}f\|_{\beta,p,q} = \|f\|_{\alpha,p,q}$, we have the following consequences.

THEOREM 4. *Let D be a bounded symmetric domain. Suppose $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, and α, β are real. Then $J^{N(\beta-\alpha)}$ is an isometric isomorphism of $H\Lambda(\alpha, p, q)$ onto $H\Lambda(\beta, p, q)$, and of $H\lambda(\alpha, p, \infty)$ onto $H\lambda(\beta, p, q)$.*

THEOREM 5. *Let D be a bounded symmetric domain. Suppose that $0 < p \leqslant \infty$, $0 < q < s < \infty$, α, β are real and $\alpha < \beta$. Then*

$$\begin{aligned} H\Lambda(\beta, p, \infty) \subset H\Lambda(\alpha, p, q) &\subset H\Lambda(\alpha, p, s) \\ &\subset H\lambda(\alpha, p, \infty) \subset H\Lambda(\alpha, p, \infty). \end{aligned}$$

Proof. $H\Lambda(\alpha, p, q) \subset H\Lambda(\alpha, p, s) \subset H\lambda(\alpha, p, \infty)$ follows from Theorem A; and $H\lambda(\alpha, p, \infty) \subset H\Lambda(\alpha, p, \infty)$ follows from the definition. Theorem D gives

$$\begin{aligned} N_{p,q,1}(J^{N(-\alpha-1)}f) &= N_{p,q,1}(J^{N(\beta-\alpha+1-1)}(J^{N(-\beta-1)}f)) \\ &\leqslant BN_{p,q,\beta-\alpha+1}(J^{N(-\beta-1)}f). \end{aligned}$$

That is,

$$\begin{aligned} N^{q}_{p,q,1}(J^{N(-\alpha-1)}f) &\leqslant B \int_0^1 (1-r)^{Nq(\beta-\alpha+1)-1} M^q_p(J^{N(-\beta-1)}f, r)\, dr \\ &\leqslant B \sup_{0<r<1} (1-r)^{Nq} M^q_p(J^{N(-\beta-1)}f, r) \\ &\quad \times \int_0^1 (1-r)^{Nq(\beta-\alpha)-1}\, dr. \end{aligned}$$

Since $Nq(\beta-\alpha)-1 > -1$, the integral converges, hence $H\Lambda(\beta, p, \infty) \subset H\Lambda(\alpha, p, q)$.

THEOREM 6. *Suppose that* $0 < p < s \leqslant \infty$, $0 < q \leqslant \infty$, *and* α *is real. Then*

$$HA(\alpha, s, q) \subset HA(\alpha, p, q) \subset HA(\alpha - 1/p + 1/s, s, q)$$

and

$$H\lambda(\alpha, s, \infty) \subset H\lambda(\alpha, p, \infty) \subset H\lambda(\alpha - 1/p + 1/s, s, \infty).$$

Proof. Since $M_p(J^{N(-\alpha-1)}f, r) \leqslant BM_s(J^{N(-\alpha-1)}f, r)$ if $p < s$, hence $HA(\alpha, s, q) \subset HA(\alpha, p, q)$, and $H\lambda(\alpha, s, \infty) \subset H\lambda(\alpha, p, \infty)$. Now

$$N_{s,q,1}(J^{N(-\alpha+1/p-1/s-1)}f) = N_{s,q,1}(J^{N(\delta+1-1)}(J^{N(-\alpha-1)}f)),$$

where $\delta = 1/p - 1/s > 0$. Then by Theorem D and Theorem A with f replaced $J^{N(-\alpha-1)}f$ we have $HA(\alpha, p, q) \subset HA(\alpha - 1/p + 1/s, s, q)$.

Since $J^{N(-\alpha-1)}f_r \in H^p$ for $r \in (0, 1)$, and $J^{N(-\alpha+1/p-1/s-1)}f_r = J^{N\delta}(J^{N(-\alpha-1)}f_r)$ by Theorem E with $\beta = \delta = 1/p - 1/s > 0$,

$$\|J^{N(-\alpha+1/p-1/s-1)}f_r\|_s \leqslant B\,\|J^{N(-\alpha-1)}f_r\|_p.$$

This implies $H\lambda(\alpha, p, \infty) \subset H\lambda(\alpha - 1/p + 1/s, s, \infty)$.

The next theorem gives an inclusion relation between the space $HA(\alpha, p, q)$ and the space H^p.

THEOREM 7. *Let D be a bounded symmetric domain. Suppose p, q, s satisfy one of* (a)–(c) *of Theorem* C. *Then* $HA(\delta, p, q) \subset H^s$, *where* $\delta = 1/p - 1/s$. *Furthermore,* $HA(\delta, p, q)$ *is dense in* H^s.

Proof. From Theorem C and Theorem D for $\alpha > 0$ we have

$$\|f\|_s \leqslant BN_{p,q,\alpha}(J^{N(-\alpha-\delta)}f) \leqslant B\,\|f\|_{\delta,p,q}.$$

This means that $HA(\delta, p, q) \subset H^s$. From (i) of Theorem 2, if $f \in H^s$ then $f_r \in HA(\delta, p, q)$ for $r \in (0, 1)$; also $f_r \to f$ in H^s as $r \to 1^-$ [2, Theorem 3]. Hence $HA(\delta, p, q)$ is dense in H^s.

THEOREM 8. *Let D be a bounded symmetric domain. Suppose* $0 < p < s \leqslant \infty$, $p \leqslant q \leqslant \infty$, $\delta = 1/p - 1/s$. *Then* $H^p \subset HA(-\delta, s, q)$. *Furthermore,* H^p *is dense in* $HA(-\delta, s, q)$.

Proof. Let $f \in H^p$. From Theorem D and Theorem B

$$\|f\|_{-\delta,s,q} \leqslant BN_{s,q,\delta}(f) \leqslant B\,\|f\|_p.$$

Hence $H^p \subset HA(-\delta, s, q)$. By (ii) of Theorem 2, if $f \in HA(-\delta, p, q)$ then $f_r \to f$ in $HA(-\delta, p, q)$ as $r \to 1^-$. Since $f_r \in H^p$ for $r \in (0, 1)$, H^p is dense in $HA(-\delta, s, q)$.

EXAMPLE. The Szegö kernel for bounded symmetric domains D is given by

$$S_\zeta(z) = S(z, \bar{\zeta}) = \sum_{k,\nu} \phi_{k\nu}(z)\,\overline{\phi_{k\nu}(\zeta)}.$$

For fixed $\zeta \in b$, S_ζ is a holomorphic function on D. In [15, Theorem 3] Mitchell proved $S_\zeta \in B_{p,q}$ for $0 < p < 1$ and $q \geqslant 2$, but $S_\zeta \notin H^p$ over the classical space $R_I(2, 2)$ if $\frac{1}{2} \leqslant p < 1$ and $S_\zeta \notin H^p$ over the classical space $R_{II}(2)$ if $\frac{2}{3} < p < 1$. The space $B_{p,q}$ is the space $HA(1/q - 1/p, q, q)$ and by Theorem 8, H^p is dense in $HA(1/q - 1/p, q, q)$.

The properties of the Szegö kernel are particularly important in the latter sections when we discuss the dual space of $HA(\alpha, p, q)$.

3. CLASSICAL LIPSCHITZ SPACES

In this section we give a definition of the classical Lipschitz spaces A_α $(0 < \alpha \leqslant 1)$ for functions of several complex variables defined on bounded symmetric domains D in C^N, and we prove that the space A_α is equivalent to the space $HA(\alpha/N, \infty, \infty)$.

A function f, holomorphic on D, with Bergman–Šilov boundary b, and continuous on $D \cup b$, belongs to Lipschitz class A_α $(0 < \alpha \leqslant 1)$ if it satisfies the Lipschitz condition

$$|f(te^{ih}) - f(t)| \leqslant C\,|h|^\alpha \qquad \text{as} \quad h \to 0, \tag{23}$$

where $t \in b$, h is real and C is independent of t and h. Since b is circular, condition (23) is the same as

$$|f(te^{i(\theta+h)}) - f(te^{i\theta})| \leqslant C\,|h|^\alpha \qquad \text{as} \quad h \to 0. \tag{24}$$

If $N = 1$ and D is the unit disc, Hardy and Littlewood [5, Theorem 5.1] proved that $f \in A_\alpha$ if and only if

$$|f'(z)| = O((1-r)^{\alpha-1}), \qquad |z| = r. \tag{25}$$

LEMMA 2. *Let D be a bounded symmetric domain in C^N. If $f \in HA(\alpha/N, \infty, \infty)$ $(0 < \alpha \leqslant 1)$, then f has a continuous extension to $D \cup b$ (that is, f is continuous on $D \cup b$).*

Proof. Let $\beta = \alpha - \alpha/N > 0$; then by Theorem D we have

$$N_{\infty,\infty,\beta}(J^{-N\alpha}f) \leqslant B N_{\infty,\infty,1}(J^{N(\beta-1)}(J^{-N\alpha}f)) = B\,\|f\|_{\alpha/N,\infty,\infty} < \infty.$$

Now

$$N_{\infty,1,\alpha}(J^{-N\alpha}f) \leqslant \sup_{0 \leqslant r<1} (1-r)^{N\alpha-\alpha} M_\infty(J^{-N\alpha}f, r) \int_0^1 (1-r)^{\alpha-1}\, dr.$$

Since $\alpha - 1 > -1$, the integral converges, and $N\alpha - \alpha = N\beta$. Hence

$$N_{\infty,1,\alpha}(J^{-N\alpha}f) \leqslant BN_{\infty,\infty,\beta}(J^{-N\alpha}f) < \infty.$$

Since $N_{\infty,1,\alpha}(J^{-N\alpha}f) < \infty$, applying (b) of Theorem C with $p = \infty$, $q = 1$, then f has a continuous extension to the Bergman–Šilov boundary b of D; that is, f is continuous on $D \cup b$.

Next we generalize the theorem of Hardy and Littlewood [5, Theorem 5.1] to bounded symmetric domains D.

THEOREM 9. *The function f belongs to Lipschitz class Λ_α $(0 < \alpha < 1)$, if and only if*

$$|J^{-1}f(rt)| \leqslant C(1-r)^{\alpha-1}, \qquad \textit{for all} \quad t \in b. \tag{26}$$

Proof. The proof uses partial function techniques. Let $f_t(w) = f(tw)$, $t \in b$, $w \in \mathbb{C}^1$ and $|w| \leqslant 1$. If $|w| < 1$, $tw \in D$; if $|w| = 1$, $tw \in b$. First we assume $f \in \Lambda_\alpha$; then f is holomorphic on D and continuous on $D \cup b$. Hence f_t is holomorphic on $|w| < 1$ and continuous on $|w| \leqslant 1$ for all $t \in b$, and it satisfies the Lipschitz condition for one complex variable. By the Hardy and Littlewood result (25)

$$|f'_t(re^{i\theta})| \leqslant C(1-r)^{\alpha-1} \qquad \text{for all} \quad t \in b. \tag{27}$$

By the definition and the proof of [5, Theorem 5.1] or [14, Theorem 6], C in (27) is independent of t.

By (7) with $m = 1$, $(J^{-1}f)_t(w) = J^{-1}f_t(w) = (wf_t(w))' = wf'_t(w) + f_t(w)$. Then for $w = re^{i\theta}$

$$\begin{aligned} |(J^{-1}f)_t(re^{i\theta})| &\leqslant |f'_t(re^{i\theta})| + |f_t(re^{i\theta})| \\ &\leqslant C(1-r)^{\alpha-1}, \end{aligned}$$

for all $t \in b$. Since $f_t(w)$ is continuous on $|w| \leqslant 1$, and hence $|f_t(w)|$ is bounded, the result follows from (27). Hence by the circularity of b we get

$$|J^{-1}f(rt')| = |J^{-1}f(rt'e^{i\theta})| \leqslant C(1-r)^{\alpha-1}, \qquad \text{for all} \quad t' \in b.$$

Conversely, assume condition (26) holds. Since $J^{-1}f_r \in H^p(D)$ for $r \in (0, 1)$, by Theorem F with $p = \infty$,

$$M_\infty(J^{-\alpha-N}f_r, r) = M_\infty(J^{-\alpha-N+1}(J^{-1}f_r), r) \leqslant B(1-r)^{-\alpha-N+1} \|J^{-1}f_r\|_\infty,$$

and by (26)

$$M_\infty(J^{-\alpha-N}f_r, r) \leqslant B(1-r)^{-\alpha-N+1+\alpha-1} = B(1-r)^{-N}. \tag{28}$$

By definition of $HA(\alpha/N, \infty, \infty)$

$$\begin{aligned}\|f\|_{\alpha/N,\infty,\infty} &= \sup_{0\leqslant r<1} (1-r^2)^N M_\infty(J^{-\alpha-N}f, r^2)\\ &\leqslant 2^N \sup_{0\leqslant r<1} (1-r)^N M_\infty(J^{-\alpha-N}f, r^2),\end{aligned}$$

which gives $\|f\|_{\alpha/N,\infty,\infty} \leqslant B$ by (28). Hence $f \in HA(\alpha/N, \infty, \infty)$, and f is holomorphic on D and continuous on $D \cup b$ by Lemma 2.

From (26) and the circularity of b we have

$$|J^{-1}f_t(re^{i\theta})| \leqslant C(1-r)^{\alpha-1}, \qquad \text{for all} \quad t \in b.$$

By (7) with $m=1$ and the fact that $f_t(w)$ is bounded on $D \cup b$, the last inequality implies

$$|f_t'(re^{i\theta})| \leqslant C(1-r)^{\alpha-1}, \qquad \text{for all} \quad t \in b.$$

Hence by the theorem for $N=1, f_t \in \Lambda_\alpha$ and

$$|f_t(e^{i(\theta+h)}) - f_t(e^{i\theta})| \leqslant C\,|h|^\alpha \qquad \text{as} \quad h \to 0,$$

where C is independent of t. Hence $f \in \Lambda_\alpha$.

THEOREM 10. *$\Lambda_\alpha = HA(\alpha/N, \infty, \infty)$ for $0 < \alpha < 1$.*

Proof. If $f \in \Lambda_\alpha$, f satisfies condition (26) by Theorem 9, and hence $f \in HA(\alpha/N, \infty, \infty)$. We have proved that $\Lambda_\alpha \subset HA(\alpha/N, \infty, \infty)$. Conversely, if $f \in HA(\alpha/N, \infty, \infty)$, then f is holomorphic on D and continuous on $D \cup b$ by Lemma 2. By Theorem D

$$\begin{aligned}N_{\infty,\infty,(1-\alpha)/N}(J^{-1}f) &\leqslant BN_{\infty,\infty,1}(J^{N((1-\alpha)/N-1)}(J^{-1}f))\\ &= B\,\|f\|_{\alpha/N,\infty,\infty}.\end{aligned}$$

Thus

$$\sup_{0\leqslant r<1} (1-r)^{1-\alpha} M_\infty(J^{-1}f, r) \leqslant C.$$

Hence $|J^{-1}f(rt)| \leqslant C(1-r)^{\alpha-1}$, for all $t \in b$, and by Theorem 9, $f \in \Lambda_\alpha$.

4. Continuous Linear Functionals on $HA(\alpha, p, q)$

From Section 2, we know that if $\min\{p, q\} \geqslant 1$, $HA(\alpha, p, q)$ is a Banach space and if $\min\{p, q\} < 1$ it is a complete linear Hausdorff space. Although $HA(\alpha, p, q)$ may not be a Banach space for $\min\{p, q\} < 1$, its bounded linear functionals can still be defined in the usual manner. Thus a linear functional F on $HA(\alpha, p, q)$ is said to be bounded if

$$\|F\| = \sup_{\|\phi\|_{\alpha,p,q}=1} |F(\phi)| \tag{29}$$

is finite. In any normed linear space [1, Theorem 11.4] a linear functional is bounded if and only if it is continuous. Formula (29) defines a norm for bounded linear functionals of $HA(\alpha, p, q)$ and the bounded linear functionals on $HA(\alpha, p, q)$ form a Banach space for all $p > 0$, $q > 0$ [1, Theorem 11.6]. We call this space the dual space of $HA(\alpha, p, q)$, and denote it by $HA^*(\alpha, p, q)$.

4.1. *Continuous Bilinear Forms*

The holomorphic functions f, g on D have Fourier series expansions (6). Define

$$\langle f, g\rangle = \lim_{r\to 1} \sum_{k,\nu} A_{k\nu}(f)\overline{A_{k\nu}(g)}\, r^k \tag{30}$$

if the limit exists. It is obvious that $\langle\, ,\rangle$ is a bilinear form. By the homogeneity and orthonormality of the system Φ_0 on b and uniform convergence,

$$\begin{aligned}\sum_{k,\nu} A_{k\nu}(f)\overline{A_{k\nu}(g)}\, r^k &= \frac{1}{V}\int_b f(r\rho^{-1}t)\,\overline{g(\rho t)}\, ds_t \\ &= (f_{r\rho^{-1}}, g_\rho)\end{aligned} \tag{31}$$

for $0 < r < \rho < 1$ [14, Sect. 2].

Theorem 11. *Suppose that D is a bounded symmetric domain in C^N. Let $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $1/p + 1/p' = 1$, α be real, and $\Omega = HA(\alpha, p, q)$. (i) If $0 < q \leqslant 1$, let $\Lambda = HA(-\alpha, p', \infty)$. (ii) If $1 < q \leqslant \infty$, let $\Lambda = HA(-\alpha, p', q')$, $1/q + 1/q' = 1$. Then the limit in* (30) *exists for $f \in \Omega$, $g \in \Lambda$ and*

$$|\langle f, g\rangle| \leqslant B\, \|f\|_\Omega\, \|g\|_\Lambda. \tag{32}$$

Hence $\langle\, ,\rangle$ is a continuous bilinear form on $\Omega \times \Lambda$.

Proof. Let $f \in \Omega$, $g \in \Lambda$ have Fourier series expansion (6). For $r \in (0, 1)$ by the homogeneity and orthonormality of the system Φ_0 on b,

$$\sum_{k,v} (k+1)^{2N} A_{kv}(f)\overline{A_{kv}(g)}\rho^{2k}r^k = \frac{1}{V}\int_b J^{N(-\alpha-1)}f(r\rho t)\,\overline{J^{N(\alpha-1)}g(\rho t)}\,ds_t.$$

Multiply both sides by $\rho(\log 1/\rho)^{2N-1}$ and integrate from 0 to 1. By the uniform convergence of the series for fixed r in $(0, 1)$, we can interchange $\sum$ and $\int_0^1$ and by the definition of the gamma function [6, (1.5)], this gives

$$\sum_{k,v} A_{kv}(f)\overline{A_{kv}(g)}\,r^k = \frac{2^{2N}}{\Gamma(2N)}\int_0^1 \rho(\log 1/\rho)^{2N-1} \times \frac{1}{V}\int_b J^{N(-\alpha-1)}f(r\rho t)\,J^{N(\alpha-1)}g(\rho t)\,ds_t\,d\rho. \quad (33)$$

By Hölder's inequality, the inequality $r(\log 1/r)^{\alpha-1} \leqslant B(1-r)^{\alpha-1}$ for $\alpha > 0$, and the monotonicity of the mean,

$$\left|\sum_{k,v} A_{kv}(f)\overline{A_{kv}(g)}\,r^k\right| \leqslant B\int_0^1 (1-\rho)^{2N-1} M_p(J^{N(-\alpha-1)}f, \rho)\,M_{p'}(J^{N(\alpha-1)}g, \rho)\,d\rho. \quad (34)$$

(i) If $0 < q \leqslant 1$, the right-hand side of (34) is less than or equal to $B\,\|g\|_{-\alpha,p',\infty}\,\|f\|_{\alpha,p,1}$ and $\|f\|_{\alpha,p,1} \leqslant B\,\|f\|_{\alpha,p,q}$ (by Theorem A). Hence

$$\left|\sum_{k,v} A_{kv}(f)\overline{A_{kv}(g)}\,r^k\right| \leqslant B\,\|f\|_\Omega\,\|g\|_\Lambda.$$

(ii) For $1 < q < \infty$, by Hölder's inequality the right-hand side of (34) is less than or equal to $B\,\|f\|_{\alpha,p,q}\,\|g\|_{-\alpha,p',q'} = B\,\|f\|_\Omega\,\|g\|_\Lambda$. For $q = \infty$, the right-hand side of (34) is less than or equal to $B\,\|f\|_{\alpha,p,\infty}\,\|g\|_{-\alpha,p',1} = B\,\|f\|_\Omega\,\|g\|_\Lambda$.

Next, we prove the limit of (30) exists for $f \in \Omega$, $g \in \Lambda$. From (33)

$$\left|\sum_{k,v} A_{kv}(f)\overline{A_{kv}(g)}\,r^k - \sum_{k,v} A_{kv}(f)\overline{A_{kv}(g)}\,r'^k\right| \leqslant B\int_0^1 (1-\rho)^{2N-1} M_p(J^{N(-\alpha-1)}(f_r - f_{r'}), \rho)\,M_{p'}(J^{N(\alpha-1)}g, \rho)\,d\rho. \quad (35)$$

If $0 < q \leqslant 1$, by the same computation as in (i), the right side of (35) is

$$B\,\|g\|_{-\alpha,p',\infty}(\|f_r - f\|_{\alpha,p,1} + \|f_{r'} - f\|_{\alpha,p,1})$$

since $p \geqslant 1$. But $\|f\|_{\alpha,p,1} \leqslant B\,\|f\|_{\Omega}$ so that the right-hand side

$$\leqslant B\,\|g\|_{\Lambda}(\|f_r - f\|_{\Omega} + \|f_{r'} - f\|_{\Omega}). \tag{36}$$

Similarly, for $1 < q < \infty$ use the same computation as in (ii) and since $p \geqslant 1$, $q > 1$, the triangle inequality holds. Hence (ii) is true for all $p > 1$, $0 < q < \infty$. For the case $q = \infty$, $q' = 1$, in formula (33) $f(rpt)$ changes to $f(\rho t)$ and $g(\rho t)$ changes to $g(rpt)$. Then by the same computation as in (35) and (36), we have

$$\begin{aligned} &\left| \sum_{k,v} A_{kv}(f)\,\overline{A_{kv}(g)}\,r^k - \sum_{k,v} A_{kv}(f)\,\overline{A_{kv}(g)}\,r'^k \right| \\ &\qquad \leqslant B\,\|f\|_{\Omega}(\|g_r - g\|_{\Lambda} + \|g_{r'} - g\|_{\Lambda}). \end{aligned} \tag{37}$$

Now apply Theorem 2(ii) on (36) and (37). In (36) $\|f_r - f\|_{\Omega}$ and $\|f_{r'} - f\|_{\Omega}$ tend to 0 as $r, r' \to 1^-$, and similarly for $\|g_r - g\|_{\Lambda}$ and $\|g_{r'} - g\|_{\Lambda}$ in (37). Hence we have proved $\langle f, g\rangle$ is bounded and

$$\left| \sum_{k,v} A_{kv}(f)\,\overline{A_{kv}(g)}\,r^k - \sum_{k,v} A_{kv}(f)\,\overline{A_{kv}(g)}\,r'^k \right| \to 0$$

as $r, r' \to 1^-$. By the Cauchy criterion the limit (30) exists for $f \in \Omega$, $g \in \Lambda$.

THEOREM 12. *Suppose that D is a bounded symmetric domain in C^N. Let $0 < p \leqslant 1$, $0 < q \leqslant \infty$, α be real and $\Omega = HA(\alpha, p, q)$. (i) If $0 < q \leqslant 1$, let $\Gamma = HA(1/p - \alpha, \infty, \infty)$. (ii) If $1 < q \leqslant \infty$, let $\Gamma = HA(1/p - 1 - \alpha, \infty, q')$. Then the limit in (30) exists for $f \in \Omega$, $g \in \Gamma$, and*

$$|\langle f, g\rangle| \leqslant B\,\|f\|_{\Omega}\,\|g\|_{\Lambda}. \tag{38}$$

Hence $\langle\, ,\,\rangle$ is a continuous bilinear form on $\Omega \times \Gamma$.

Proof. Let $f \in \Omega$, $g \in \Gamma$ have Fourier series expansion (6). Let $\beta = \alpha + 1 - 1/p$, $-\beta = 1/p - 1 - \alpha$. By (37) and Hölder's inequality with $p = 1$, $p' = \infty$,

$$\begin{aligned} &\left| \sum_{k,v} A_{kv}(f)\,\overline{A_{kv}(g)}\,r^k \right| \\ &\qquad \leqslant B \int_0^1 (1-\rho)^{2N-1}\, M_1(J^{N(-\beta-1)}f, \rho)\, M_\infty(J^{N(\beta-1)}g, \rho)\, d\rho. \end{aligned} \tag{39}$$

The results in Theorem 12 therefore follow by using (39), Theorems 5 and 6, and the same argument as in the proof of Theorem 11.

4.2. *Representation of Continuous Linear Functionals on $H\Lambda(\alpha, p, q)$*

We need the following lemmas for the proofs of Theorems 13 and 14. First, we prove that the complete orthonormal system $\Phi_0 = \{\phi_{k\nu}, k = 0, 1, 2,...; \nu = 1, 2,..., m_k\}$ is contained in the space $H\Lambda(\alpha, p, q)$.

LEMMA 3. *Let $0 < p \leqslant \infty$, $0 < q \leqslant \infty$, α be real. Then $\phi_{k\nu} \in H^p(D)$ and to $H \wedge (\alpha, p, q)$ for all $k = 0, 1, 2,...; \nu = 1,..., m_k$ and*

$$\|\phi_{k\nu}\|_{\alpha,p,q} \leqslant B(k+1)^{N(\alpha+1)} \|\phi_{k\nu}\|_p. \tag{40}$$

Proof. Since $\phi_{k\nu}$ is a homogeneous polynomial on D, it is holomorphic on D and bounded on $\bar{D}$. Hence $\phi_{k\nu} \in H^p(D)$. Since $J^{N(-\alpha-1)}\phi_{k\nu} = (k+1)^{N(\alpha+1)}\phi_{k\nu}$, for $p < \infty$, we have

$$\begin{aligned}\|\phi_{k\nu}\|_{\alpha,p,q} &\leqslant (k+1)^{N(\alpha+1)} \left[\sup_{0 \leqslant r < 1} M_p^q(\phi_{k\nu}, r) \int_0^1 (1-r)^{Nq-1}\, dr\right]^{1/q} \\ &\leqslant B(k+1)^{N(\alpha+1)} \|\phi_{k\nu}\|_p.\end{aligned}$$

Similarly for $q = \infty$.

LEMMA 4. *Let $0 < p \leqslant \infty$, $0 < q < \infty$; suppose that $F \in H\Lambda^*(\alpha, p, q)$ is a bounded linear functional on $H\Lambda(\alpha, p, q)$. Let*

$$g(z) = \sum_{k,\nu} \overline{F(\phi_{k\nu})}\, \phi_{k\nu}(z). \tag{41}$$

Then (i) *g is a holomorphic function on D.* (ii) *$F(f) = \langle f, g\rangle$ for all $f \in H\Lambda(\alpha, p, q)$.*

Proof. First we prove that the series in (41) converges uniformly on compact subsets of D. By (40)

$$\sum_{k,\nu} |\overline{F(\phi_{k\nu})}\, \phi_{k\nu}(z)| \leqslant B\, \|F\| \sum_{k,\nu} (k+1)^{N(\alpha+1)} \|\phi_{k\nu}\|_p\, |\phi_{k\nu}(z)|. \tag{42}$$

By [14, Theorem 1], $\sum_{k,\nu} \|\phi_{k\nu}\|_p\, |\phi_{k\nu}(z)|$ is uniformly convergent on compact subsets of D, so that the function $h(z) = \sum_{k,\nu} \|\phi_{k\nu}\|_p\, \phi_{k\nu}(z)$ is holomorphic on D. Since $J^{N(-\alpha-1)}h(z)$ is also holomorphic on D, the series $\sum_{k,\nu} (k+1)^{N(\alpha+1)} \|\phi_{k\nu}\|_p\, |\phi_{k\nu}(z)|$ converges uniformly on compact subsets of D. Hence by (42) the series (41) converges uniformly on compact subsets of D so that g is holomorphic on D.

Let $f \in HA(\alpha, p, q)$. Since f_r is the uniform limit on b of partial sums of its series expansion, and F is a continuous linear functional

$$F(f_r) = \sum_{k,v} A_{kv}(f)\, F(\phi_{kv})\, r^k.$$

By Theorem 2(ii), $f_r \to f$ in $HA(\alpha, p, q)$ as $r \to 1^-$ and by the continuity of F,

$$F(f) = \lim_{r \to 1} \sum_{k,v} A_{kv}(f)\, F(\phi_{kv})\, r^k = \langle f, g \rangle.$$

LEMMA 5. *The Szegö kernel $S_{\bar\zeta}(z) = S(z, \bar\zeta)$, $z \in D$, $\zeta \in D \cup b$, of a bounded symmetric domain D belongs to $HA(\alpha, p, q)$ for $\alpha < -1 + 1/p$, $p \geqslant 2$, and to $HA(\alpha, p, q)$ for $0 < p < 2$, $\alpha < -\frac{1}{2}$.*

Proof. Let $p \geqslant 2$, $\alpha < -1 + 1/p < 0$, and let $\beta = -\alpha > 0$. By Theorem D

$$\| S_{\bar\zeta} \|^q_{\alpha,p,q} \leqslant B N^q_{p,q,-\alpha}(S_{\bar\zeta}) = B \int_0^1 (1-r)^{Nq\beta - 1} M^q_p(S_{\bar\zeta}, r)\, dr. \tag{43}$$

Since $S(z, \bar\zeta)$ is holomorphic on $D \times D$ and continuous on $D \times D \cup b$, by [12, Theorems 4.6.1 and 4.5.1] and the maximum principle

$$\begin{aligned} M^2_2(S_{\bar\zeta}, r) &= \frac{1}{V} \int_b S(\zeta, r\bar t)\, S(rt, \bar\zeta)\, ds_t = S(r\zeta, r\bar\zeta) \\ &= \sum_{k,v} |\phi_{kv}(\zeta)|^2\, r^2 \leqslant \frac{1}{V} (1-r)^{-N} \end{aligned}$$

and

$$\max_{t \in b} S(rt, \bar\zeta) = \frac{1}{V} (1-r)^{-N}.$$

Hence

$$\begin{aligned} M^p_p(S_{\bar\zeta}, r) &\leqslant \max_{t \in b} |S_{\bar\zeta}(rt)|^{p-2} \frac{1}{V} \int_b |S_{\bar\zeta}(rt)|^2\, ds_t \\ &\leqslant B(1-r)^{-N(p-1)}. \end{aligned} \tag{44}$$

Substituting (44) into (43), we have

$$\| S_\zeta \|^q_{\alpha,p,q} \leqslant B \int_0^1 (1-r)^{Nq(\beta - 1 + 1/p) - 1}\, dr < \infty,$$

since $\beta - 1 + 1/p = -\alpha - 1 + 1/p > 0$.

For $0 < p < 2$, $\alpha < -\frac{1}{2}$, let $\beta = -\alpha > \frac{1}{2}$. Since $M_p(S_{\bar{\zeta}}, r) \leqslant M_2(S_{\bar{\zeta}}, r)$, by (43) and (44)

$$\| S_\zeta \|_{\alpha,p,q}^q \leqslant B \int_0^1 (1-r)^{Nq\beta - 1}(1-r)^{-Nq/2}\, dr < \infty.$$

From Theorems 11 and 12, and Lemmas 4 and 5 we have following two theorems for the representation of a bounded linear functional in $HA(\alpha, p, q)$.

THEOREM 13. *Suppose that D is a bounded symmetric domain in C^N. Let $1 \leqslant p \leqslant \infty$, $0 < q < \infty$ and α be real. Let $\Omega = HA(\alpha, p, q)$, $\Lambda = HA(-\alpha, p', \infty)$ if $0 < q \leqslant 1$, $\Lambda = HA(-\alpha, p', q')$ if $1 < q < \infty$. Then*

(i) *If $g \in \Lambda$ and $F_g(f) = \langle f, g\rangle$ for all $f \in \Omega$, then $F_g \in \Omega^* = HA^*(\alpha, p, q)$ and the norm $\|F_g\|_{\Omega^*} \leqslant B \|g\|_\Lambda$.*

(ii) *For each $F \in \Omega^*$ there exists a unique holomorphic function g such that $F(f) = \langle f, g\rangle$ for all $f \in \Omega$.*

(iii) *If $p \geqslant 2$, $\alpha < -1 + 1/p$, or $1 < p < 2$, $\alpha < -\frac{1}{2}$, then $g \in HA(1/p' - 1/s, p', \infty)$ for $0 < q \leqslant 1$, $s < 1$, and $g \in HA(1/p' - 1/s, p', q')$ for $1 < q < \infty$, $s < 1$.*

Proof. Part (i) follows by Theorem 11. To prove (ii), let $F \in \Omega^*$ and let

$$g(z) = \sum_{k,\nu} \overline{F(\phi_{k\nu})}\, \phi_{k\nu}(z). \tag{45}$$

By Lemma 4, g is holomorphic on D and $F(f) = \langle f, g\rangle$. To prove that g is unique it is sufficient to prove that $\langle f, g\rangle = 0$ for all $f \in \Omega$ implies $g = 0$. Since $\phi_{k\nu} \in \Omega$ by Lemma 3, $\langle \phi_{k\nu}, g\rangle = \overline{A_{k\nu}(g)} = 0$ for all k and ν. This implies that the Fourier coefficients of g are all zero, hence $g = 0$. (iii) If $p \geqslant 2$, $\alpha < -1 + 1/p$, or if $1 \leqslant p < 2$, $\alpha < -\frac{1}{2}$, by Lemma 5 the Szegö kernel $S_{\bar{\zeta}}(z) \in \Omega = HA(\alpha, p, q)$ for $\zeta \in D \cup b$. Since the series for $S(z, \bar{\zeta})$ converges uniformly for $\zeta \in D \cup b$ and z belongs to a compact subset of D, by the continuity of F and (45),

$$\overline{g(\zeta)} = F(S_{\bar{\zeta}}).$$

Hence

$$|g(\zeta)| \leqslant \|F\| \cdot \|S_{\bar{\zeta}}\|_\Omega, \qquad \text{for all} \quad \zeta \in D \cup b.$$

Hence $g \in H^\infty(D) \subset H^s(D)$ for all $s > 0$. Choose $s < 1$, since $s < p'$ by Theorem 8, $H^s(D) \subset HA(1/p' - 1/s, p', \infty)$ for all $0 < q \leqslant 1$, and $H^s(D) \subset HA(1/p' - 1/s, p', q')$ for $1 < q < \infty$. Hence (iii) follows.

THEOREM 14. *Suppose that D is a bounded symmetric domain in C^N. Let $0 < p \leqslant 1$, $0 < q < \infty$, α be real. Let $\Omega = HA(\alpha, p, q)$, $\Gamma = HA(1/p - 1 - \alpha, \infty, \infty)$ if $0 < q \leqslant 1$; $\Gamma = HA(1/p - 1 - \alpha, \infty, q')$ if $1 < q < \infty$. Then*

(i) *If $g \in \Gamma$ and $F_g(f) = \langle f, g\rangle$ for all $f \in \Omega$, then $F_g \in \Omega^*$ and the norm $\|F_g\| \leqslant B \| g\|_\Gamma$.*

(ii) *If $f \in \Omega^*$, then there exists a unique holomorphic function g such that $F(f) = \langle f, g\rangle$ for all $f \in \Omega$.*

(iii) *If $\alpha < -\frac{1}{2}$, then $g \in HA(-1/s, \infty, \infty)$ for $0 < q \leqslant 1$, $s < 1$, and $g \in HA(-1/s, \infty, q')$ for $1 < q < \infty$, $s < 1$.*

Proof. The proof is similar to the proof of Theorem 13. Part (i) follows by Theorem 12, (ii) by Lemma 4, and (iii) by Lemma 5 for $0 < p \leqslant 1$, $\alpha < -\frac{1}{2}$, and the same argument as in the proof of Theorem 13(iii).

5. EQUIVALENCE OF TWO BANACH SPACES FOR THE BALL

Let D be the ball with center 0 and radius 1 in C^N. Its Szegö kernel is

$$S(z, \bar{t}) = (1/V)(1 - zt^*)^{-N}, \qquad z \in D, \quad t \in \bar{D}, \tag{46}$$

where V is the euclidean volume of b and $zt^* = \sum_1^N z_i \bar{t}_i$ [12, 14].

THEOREM 15. *(Suppose that D is the ball in C^N.) Let $0 < p \leqslant 1$, $0 < q \leqslant 1$, and α be real, $\Omega = HA(\alpha, p, q)$, $\Gamma = HA(1/p - 1 - \alpha, \infty, \infty)$. Then*

(i) *If $g \in \Gamma$ and $F_g(f) = \langle f, g\rangle$ for all $f \in \Omega$, then $F_g \in \Omega^*$ and the norm $\|F_g\| \leqslant B \| g\|_\Gamma$.*

(ii) *Conversely, for each $F \in \Omega^*$, there exists a unique $g \in \Gamma$ such that $F(f) = \langle f, g\rangle$ for all $f \in \Omega$ and the norm $\|F\|$ satisfies $B \| g\|_\Gamma \leqslant \|F\| \leqslant B \| g\|_\Gamma$, that is, the two norms $\|F\|$ and $\| g\|_\Gamma$ are equivalent.*

(iii) *Furthermore, the mapping $g \to F_g$ is a linear homeomorphism of Γ onto Ω^*; that is, Γ and Ω^* are equivalent.*

Proof. (i) is contained in Theorem 14(i). (ii) For any $F \in \Omega^*$ we define $g(z) = \sum_{k,\nu} \overline{F(\phi_{k\nu})}\, \phi_{k\nu}(z)$. By Theorem 14(ii), g is holomorphic and unique with $F(f) = \langle f, g\rangle$ for all $f \in \Omega$. To prove $g \in \Gamma$, fixed $t_0 \in b$, the Szegö kernel $S_{\bar{t}_0}(z)$ and $J^{N(\alpha - 1/p)} S_{\bar{t}_0}(z)$ are holomorphic functions on the ball. Let

$$h(z) = J^{N(\alpha - 1/p)} S_{\bar{t}_0}(z) = \sum_{k,\nu} (k+1)^{N(1/p - \alpha)}\, \overline{\phi_{k\nu}(t_0)}\, \phi_{k\nu}(z).$$

Then by Theorem 2(i), $h_r(z) \in \Omega$, and by the continuity of F,

$$F(h_r) = J^{N(\alpha - 1/p)}\overline{g(rt_0)}.$$

Hence

$$|J^{N(\alpha - 1/p)}g(rt)| \leqslant \|F\| \cdot \|h_r\|_\Omega. \tag{47}$$

Now we compute the value $\|h_r\|_\Omega$. By Theorem 3, $h_r \in HA(\alpha, p, q)$ if and only if there exists $\gamma > 0$ such that $N_{p,q,\gamma}(J^{N(-\alpha-\gamma)}h_r) < \infty$ and

$$\|h_r\|_\Omega \leqslant BN_{p,q,\gamma}(J^{N(-\alpha-\gamma)}h_r). \tag{48}$$

Now

$$J^{N(-\alpha-\gamma)}h_r(\rho t) = J^{N(-\gamma-1/p)}S_{r\bar{t}_0}(\rho t).$$

Choose $\gamma > 0$ so that $N(\gamma + 1/p)$ is an integer and by (7) and (46),

$$J^{-N(\gamma+1/p)}S_{r\bar{t}_0}(\rho t) = \left(\frac{\partial}{\partial\rho}\rho\right)^{N(\gamma+1/p)}(V^{-1}(1-\rho rtt_0^*)^{-N}).$$

Hence

$$|J^{N(-\alpha-\gamma)}h_r(\rho t)| = O(|1-\rho rtt_0^*|^{-N-N(\gamma+1/p)}) \tag{49}$$

and

$$M_p(J^{N(-\alpha-\gamma)}h_r, \rho) \leqslant B\left[\int_b |1-\rho rtt_0^*|^{-(N+N\gamma)p-N}\,ds_t\right]^{1/p}.$$

By [14, p. 392]

$$\int_b |1-\rho rtt_0^*|^{-(N+N\gamma)p-N}\,ds_t \leqslant B(1-\rho r)^{-(N+N\gamma)p}. \tag{50}$$

Thus

$$M_p(J^{N(-\alpha-\gamma)}h_r, \rho) \leqslant B(1-\rho r)^{-(N+N\gamma)}. \tag{51}$$

Hence from (48) and (51)

$$\begin{aligned}\|h_r\|_\Omega &\leqslant B\left[\int_0^1 (1-\rho)^{Nq\gamma-1}(1-\rho r)^{-(N+N\gamma)q}\,d\rho\right]^{1/q}\\ &= B\left[r^{-Nq\gamma}\int_0^r (1-\sigma)^{-(N+N\gamma)q}(r-\sigma)^{Nq\gamma-1}\,\sigma^{1-1}\,d\sigma\right]^{1/q}\end{aligned}$$

by setting $\sigma = r\rho$. Then by [8, (15.3)]

$$\|h_r\|_\Omega \in B(1-r)^{-N}. \tag{52}$$

From (47) and (52)

$$\sup_{t_0 \in b} |J^{N(\alpha - 1/p)} g(rt_0)| \leqslant B \, \|F\| \, (1-r)^{-N}.$$

Hence

$$\|g\|_\Gamma = \sup_{0 \leqslant r < 1} (1-r)^N M_\infty (J^{N(\alpha - 1/p)} g, r) \leqslant B \, \|F\| \tag{53}$$

so that $g \in \Gamma$. By Part (i) and (53)

$$B \, \|g\|_\Gamma \leqslant \|F\| \leqslant B \, \|g\|_\Gamma. \tag{54}$$

To prove (iii), we define a linear mapping L: $g \to F_g$ from Γ to Ω^* by part (i). L is linear since $\langle f, g \rangle$ is a continuous bilinear form (Theorem 12). If $F_g = F_{g'}$, that is, $\langle f, g \rangle = \langle f, g' \rangle$ for all $f \in \Omega$ or $\langle f, g - g' \rangle = 0$ for all $f \in \Omega$, then $g = g'$, so that L is one to one. By part (ii), L is onto. By (54) both L and L^{-1} are bounded. Hence L is a linear homeomorphism and Ω^*, Γ are equivalent.

From Theorem 5 we know that $HA(\alpha, p, q) \subset HA(\alpha, p, 1)$ for $0 < p < 1$, $0 < q < 1$. But by Theorem 15 the dual spaces of $HA(\alpha, p, q)$ and $HA(\alpha, p, 1)$ are linearly homeomorphic to the same space for all $0 < q < 1$ and for fixed p between 0 and 1; that is, these two dual spaces are equivalent to $HA(1/p - 1 - \alpha, \infty, \infty)$. Hence

THEOREM 16. *Let $0 < p$, $q < 1$, α be real, $\Omega = HA(\alpha, p, q)$ and $\Lambda = HA(\alpha, p, 1)$. Then for $F \in \Lambda^*$, the restriction mapping $F \to F|_\Omega$ is a linear homeomorphism of Λ^* onto Ω^*.*

In [14, Sect. 6] Mitchell and Hahn introduced B^p spaces on bounded symmetric domains for $0 < p < 1$, and proved H^p is dense in B^p. The B^p space by definition is our $HA(1 - 1/p, 1, 1)$ space. When the domain D is the ball by Theorem 15, we know that the dual space of B^p is equivalent to the space $HA(1/p - 1, \infty, \infty)$. Next we prove that the dual space of H^p $(0 < p < 1)$ also is equivalent to $HA(1/p - 1, \infty, \infty)$. Hence for $0 < p < 1$ the dual spaces of H^p and B^p are equivalent to the same space for the ball. (See [4, Theorem 7] for the case of one complex variable.)

THEOREM 17. *Suppose that D is the ball in C^N. Let $0 < p < 1$, and $\Gamma = HA(1/p - 1, \infty, \infty)$.*

(i) *If* $g \in \Gamma$, *and* $F_g(f) = \langle f, g \rangle$ *for all* $f \in H^p$, *then* $F_g \in (H^p)^*$ *and the norm* $\|F_g\| \leqslant B \|g\|_\Gamma$.

(ii) *Conversely, for each* $F \in (H^p)^*$, *there exists a unique* $g \in \Gamma$ *such that* $F(f) = \langle f, g \rangle$ *for all* $f \in H^p$ *and the norm* $\|F\|$ *satisfies* $B \|g\|_\Gamma \leqslant \|F\| \leqslant B \|g\|_\Gamma$.

(iii) *Further, the mapping* $g \to F_g$ *is a linear homeomorphism of* Γ *onto* $(H^p)^*$; *that is,* Γ *and* $(H^p)^*$ *are equivalent.*

Proof. Let $\Omega = H\Lambda(1 - 1/p, 1, 1) = B^p$. Since $H^p \subset B^p$, by Theorem 12(i) with $p = q = 1$, $\alpha = 1 - 1/p$, $\langle f, g \rangle$ is a continuous bilinear form on $\Omega \times \Gamma$ and also a bilinear form on $H^p \times \Gamma$, and

$$|\langle f, g \rangle| \leqslant B \|f\|_\Omega \|g\|_\Gamma.$$

But by Theorems D and B

$$\|f\|_\Omega \leqslant B N_{1,1,1/p-1}(f) \leqslant B \|f\|_p.$$

Hence part (i) follows.

(ii) Let $F \in (H^p)^*$. Since $\phi_{k\nu} \in H^p$, we can define $g(z) = \sum_{k,\nu} \overline{F(\phi_{k\nu})}\, \phi_{k\nu}(z)$.

As in the proof of Lemma 4, or by [14, Theorem 1], g is holomorphic and unique with $F(f) = \langle f, g \rangle$ for all $f \in H^p$. To prove that $g \in \Gamma = H\Lambda(1/p - 1, \infty, \infty)$, by Theorem 3 it is sufficient to prove that $N_{\infty,\infty,\gamma}(J^{N(1-1/p-\gamma)} g) < \infty$ for some $\gamma > 0$.

For fixed $t_0 \in b$ the Szegö kernel $S_{\bar{t}_0}(z)$ and $J^{N(1-1/p-\gamma)} S_{\bar{t}_0}(z)$ are holomorphic functions on the ball. Set

$$h(z) = J^{N(1-1/p-\gamma)} S_{\bar{t}_0}(z) = \sum_{k,\nu} (k+1)^{N(1/p-1+\gamma)}\, \overline{\phi_{k\nu}(t_0)}\, \phi_{k\nu}(z).$$

For $r \in (0, 1)$, $h_r \in H^p$, and by the continuity of F

$$F(h_r) = J^{N(1-1/p-\gamma)} \overline{g(rt_0)}.$$

Hence

$$\begin{aligned} |J^{N(1-1/p-\gamma)} g(rt_0)| &\leqslant \|F\| \cdot \|h_r\|_p \\ &= \|F\| \cdot \|J^{N(1-1/p-\gamma)} S_{r\bar{t}_0}\|_p. \end{aligned} \tag{55}$$

Now we compute $\|J^{N(1-1/p-\gamma)} S_{r\bar{t}_0}\|_p$. Choose $\gamma > 0$ so that $N(1 - 1/p - \gamma)$ is a negative integer; by the same argument as in (49) and (50)

$$J^{N(1-1/p-\gamma)} S_{r\bar{t}_0}(\rho t) = O(|1 - \rho r t t_0^*|^{-N(1/p+\gamma)}),$$

and

$$M_p(J^{N(1-1/p-\gamma)}S_{r\bar{t}_0}, \rho) \leqslant B(1-\rho r)^{-N\gamma}. \tag{56}$$

Since $(1-\rho r)^{-N\gamma} \leqslant (1-r)^{-N\gamma}$ if $0 < \rho < 1$, by (56)

$$\|J^{N(1-1/p-\gamma)}S_{r\bar{t}_0}\|_p \leqslant B(1-r)^{-N\gamma}. \tag{57}$$

By (55) and (57)

$$M_\infty(J^{N(1-1/p-\gamma)}g, r) \leqslant B\,\|F\|\,(1-r)^{-N\gamma},$$

so that

$$\begin{aligned} N_{\infty,\infty,\gamma}(J^{N(1-1/p-\gamma)}g) &\leqslant \sup_{0 \leqslant r < 1} (1-r)^{N\gamma} B\,\|F\|\,(1-r)^{-N\gamma} \\ &= B\,\|F\|. \end{aligned}$$

Thus $g \in \Gamma$, and $B\,\|g\|_\Gamma \leqslant \|F\| \leqslant B\,\|g\|_\Gamma$.

The proof of (iii) follows from (i) and (ii) in the same ways as (iii) follows from (i) and (ii) in Theorem 15.

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